Margulis Number for Hyperbolic 3-manifolds

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Abstract

Margulis number provides useful information about the thin part of a hyperbolic manifold. By Margulis Lemma, we know that the Margulis 3-constant exists but this number still not yet be discovered.

This thesis is an exposition of the study of Margulis number according to Shalen and Culler. We will demonstrate the largest known Margulis number for closed orientable hyperbolic 3-manifold whose first Betti number is at least 3, which is proved by Shalen and Culler in 1992.

摘要

Margulis數值可以提供關於雙曲流形的細部份的資料。 根據Margulis引理，我們知道3維的Margulis常數是實際存在的， 但是還沒有人知道它的真確數值。

在本論文中，我們最主要研習Shalen和Culler的研究結果。 我們會展示Shalen和Culler在1992年找出的對於首個貝蒂數至少為3的3維可定向雙曲閉流形的已知最大的Margulis數值。

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Contents

Chapter 1

Introduction

Margulis number can be used to decompose the thin part of a hyperbolic manifold into disjoint cusps and solid tori. Besides, we can also apply the knowledge of Margulis number on the thick part to get a volume estimation of the hyperbolic manifold but we will not go through the detail here.

Throughout the years, many mathematicians such as F.Gehring, G.Martin and R. Meyerhoff gave out some lower bound of the volume of an arbitrary closed orientable hyperbolic 3-manifold and C.Hodgson, J.Weeks have shown that $\log 3$ is not Margulis constant, i.e. not a Margulis number for all hyperbolic 3-manifold. Recently, Shalen and Culler have finished a lot of works about finding a Margulis number of different kind of hyperbolic manifolds. We will exposition their method used for closed orientable hyperbolic 3-manifold whose first Betti number is at least 3.

The main result that we will present is due to Shalen and Culler:

Theorem 6.3. Let M be closed, orientable hyperbolic 3-manifold such that $rank(H_1(M;\mathbb{Q})) \geq 3$. Then $\log 3$ is a Margulis number for M.

In Chapter 2, we introduce basic knowledge of hyperbolic manifolds, especially those related to Margulis number so that our discussion will be smoother later. Reference of such knowledge can be found in Beardon [6], Fenchel [15], Matsuzaki and Taniguchi [21].

In Chapter 3, a family of measures so-called conformal densities for $\mathbb{S}_\infty$ is defined. Then the relation of conformal densities and superharmonic functions is studied. At the end, we give a description of the important result Theorem 3.2 that the conformal densities will be a constant multiple of the area density whenever every Γ-invariant positive superharmonic function is constant. This Theorem provides us some information about whether the hypothesis of Theorem 4.3 holds.

In Chapter 4, by making use of Patterson achievement and some extension of Nicholls, Theorem 4.3 which is closely related to the main result can be obtained.

Theorem 4.3. Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$. Suppose $\Gamma = \langle \xi, \eta \rangle$ is discrete and free on ξ and η. Suppose every Γ-invariant conformal densities is a constant multiple of the area density. Then $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, for all z in $\mathbb{H}^3$.

In Chapter 5, some conditions which ensure the hypothesis of Theorem 3.2 hold are found by Shalen and Culler with the knowledge about Bonahon surfaces and grided surface. As a result, we can apply Theorem 4.3 when these conditions hold.

In Chapter 6, we go back to our concerning situtation and consider mainly for torsion-free Kleinian group without parabolics. By using previous results, Theorem 6.1 can be proved.

Theorem 6.1. Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$ with $\xi\eta \neq \eta\xi$. Suppose $\langle \xi, \eta \rangle$ is torsion free, discrete, topologically tame, noncocompact and contains no parabolics. Then $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$.

Then our main result Theorem 6.3 can be proved by applying Theorem 6.1.

Chapter 2

Elementary properties and notations of Hyperbolic space

In this chapter, we write down some basic knowledge about hyperbolic space, including the definition of thin part and Margulis number of a hyperbolic manifold.

Here we use:

$\mathbb{S}_\infty$ to represent the sphere at infinity;

$\overline{\mathbb{H}}^3$ to represent the compactification of $\mathbb{H}^3$ with boundary $\mathbb{S}_\infty$;

$d(\cdot,\cdot)$ to represent the hyperbolic distance;

$nbhd_r(K)$ to represent the set $\{x \in \mathbb{H}^3 : d(x, K) < r\}$.

Definition 2.1. *Let $K \subset \overline{\mathbb{H}}^3$, $K \neq \emptyset$.*

K is called convex if every line segment with end points in K is contained in K.

Let $X \subset \overline{\mathbb{H}}^3$, we define the convex *hull* of X as:

Definition 2.2. $\overline{hull}(X) := \bigcap\limits_{X \subset K \subset \overline{\mathbb{H}}^3, K\, convex} K$ *and* $hull(X) := \overline{hull}(X) \cap \mathbb{H}^3$.

Definition 2.3. *Let $\gamma \in Isom^+(\mathbb{H}^3)$, $\gamma \neq 1$, $\epsilon > 0$.*

$C_\epsilon(\gamma) := \{x \in \mathbb{H}^3 : d(x, \gamma(x)) \leq \epsilon\}$.

Let γ be a loxodromic element in $Isom^+(\mathbb{H}^3)$, i.e. γ has exactly two fixed points in $\mathbb{S}_\infty$, A_γ be the geodesic joining the fixed points of γ, i.e. the axis of γ. Then we have the following proposition.

Proposition 2.1. *Let $\gamma \in Isom^+(\mathbb{H}^3)$ be loxodromic with axis A_γ.*

Then $\exists$ $f_\gamma : [0, \infty) \to [0, \infty)$ monotonically increasing continuous such that:

$$d(z, \gamma(z)) = f_\gamma(d(z, A_\gamma)),\ \forall z \in \mathbb{H}^3$$

Proof. Consider upper half-space model, by taking conjugation, we may assume $\gamma((x, t)) = kx + |k|t$, $|k| \neq 1$. Then we have $d(w, \gamma(w)) = \log(|k|) = k_1$, $\forall w \in A_\gamma$. Now let $z = (x, t) \in \mathbb{H}^3$. Then the closest point of z in A_γ is $w = (0, (|x|^2 + t^2)^{\frac{1}{2}})$.

As a result, we have:

$$cosh(d(z,w)) = \sqrt{\frac{|x|^2}{t^2} + 1}$$

and

$$cosh(d(z,\gamma(z))) = \frac{|k-1|^2}{2|k|}(\frac{|x|^2}{t^2})+(\frac{|k|^2+1}{2|k|}) = \frac{|k-1|^2}{2|k|}(cosh^2(d(z,w))-1)+(\frac{|k|^2+1}{2|k|})$$

By the monotonically increasing property of $cosh$, we can get the result. $\square$

Remark. We use $length(\gamma)$ to represent the translation distance $f_\gamma(0)$ on A_γ.

By Proposition (2.1), when γ is loxodromic element, we have:

$$C_\epsilon(\gamma) = \begin{cases} \emptyset & \text{if } \epsilon < length(\gamma) \\ nbhd_{f_\gamma^{-1}(\epsilon)}(A_\gamma) & \text{if } \epsilon \geq length(\gamma) \end{cases}$$

Let Γ be a Kleinian group, where Kleinian group is defined as nonelementary discrete subgroup of $PSL_2(\mathbb{C})$ throughout this paper, then we use the following conventions:

$\Lambda(\Gamma)$ to represent the limit set of Γ.

In addition if Γ is torsion free, we use:

$M(\Gamma)$ to represent the complete hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$;

$N(\Gamma)$ to represent $(nbhd_1(hull(\Lambda(\Gamma))))/\Gamma$.

By definition, we have $N(\Gamma)$ is a contractible, Γ-invariant 3-manifold with boundary.

Below we prove a proposition which is useful for the decomposition of the thin part of a hyperbolic manifold.

Proposition 2.2. *Let H be a maximal abelian subgroup of a torsion free Kleinian group Γ. Then we have either:*

(i) All the elements of H are loxodromic with same axis.

(ii) All the elements of H are parabolic with same fixed point.

In case (i) H is cyclic.

In case(ii) H is cyclic or free abelian with rank 2.

H is called cuspidal subgroup of Γ if case(ii) occur.

Proof. As Γ is a torsion free Kleinian group, H does not contain any elliptic element. As H is abelian, all of the elements in H have same fixed points (By Theorem 4.3.6, [6]). If the number of fixed points is 1, then all the elements are parabolic. If the number of fixed points is 2, then all the elements are loxodromic. As a Kleinian group, we must have H is cyclic if H is purely loxodromic and H is cyclic or free abelian with rank 2 if H is purely parabolic as H is discrete.

□

Then we write down some definitions about the thin part of a hyperbolic manifold.

Definition 2.4. *Let $p \in M(\Gamma)$, where Γ is a torsion free Kleinian group.*

$$short(p) := \begin{cases} \inf\limits_{1 \neq \gamma \in \Gamma} d(x, \gamma(x)) & \text{if } \Gamma \neq \{1\} \\ \infty & \text{if } \Gamma = \{1\} \end{cases}$$

Here $short(p)$ is the length of the shortest nontrivial loop based at p.

Definition 2.5. *Let $I \subset (0, \infty)$, $\epsilon > 0$.*

$M_I(\Gamma) := \{p \in M(\Gamma) : short(p) \in I\}$.

$M_{(0,\epsilon]}(\Gamma)$ *is called ϵ-thin part of $M(\Gamma)$.*

Let $\widetilde{M}_{(0,\epsilon]}(\Gamma)$ be the preimage of $M_{(0,\epsilon]}(\Gamma)$ in $\mathbb{H}^3$. Then we have:

$$\widetilde{M}_{(0,\epsilon]}(\Gamma) = \{x \in \mathbb{H}^3 : d(x, \gamma(x)) \leq \epsilon \text{ for some } \gamma \in \Gamma - \{1\}\} = \bigcup_{1 \neq \gamma \in \Gamma} C_\epsilon(\gamma)$$

Let H be a maximal abelian subgroup of Γ, then $\{C_\epsilon(\gamma)\}$ is totally ordered and we denote $\{C_\epsilon(H)\}$ to be the maximal set. Then we have

$$\widetilde{M}_{(0,\epsilon]}(\Gamma) = \bigcup_{H \leq \Gamma,\ maximal\ abelian} C_\epsilon(H)$$

With the above setting, we can define what Margulis number is.

Definition 2.6. *$\epsilon > 0$ is called a Margulis number for $M(\Gamma)$ if $C_\epsilon(H_1) \cap C_\epsilon(H_2) = \emptyset$, $\forall H_1, H_2 \leq \Gamma$ maximal abelian, $H_1 \neq H_2$.*

Equivalently, we have:

Definition 2.7. *$\epsilon > 0$ is called a Margulis number for $M(\Gamma)$ if $max(d(x, \xi(x)), d(x, \eta(x))) \geq \epsilon$, $\forall x \in \mathbb{H}^3, \xi, \eta \in \Gamma, \xi\eta \neq \eta\xi$.*

Definition 2.8. *$\epsilon > 0$ is called 3-dimensional Margulis constant if ϵ is the largest number which is a Margulis number for all hyperbolic 3-manifold.*

Remark. The existance of 3-dimensional Margulis constant is provided by Margulis Lemma. However this number is not known yet.

When ϵ is a Margulis number, then $M_{(0,\epsilon]}(\Gamma) = \bigcup_{H \leq \Gamma,\ maximal\ abelian} C_\epsilon(H)/H$. If H is in case(i) of Proposition 2.2, then $C_\epsilon(H)/H$ is either a solid torus or a empty set from the point of view of Proposition 2.1. If H is in case(ii), then $C_\epsilon(H)/H$ is a cusp.

Below we define some notations which will appear in the latter chapters.

Definition 2.9. $M^c_{(0,\epsilon]}(\Gamma) := \bigcup_{H \leq \Gamma,\ cuspidal} C_\epsilon(H)/H;$

$M^c_{\{\epsilon\}}(\Gamma) := \partial M^c_{(0,\epsilon]};$

$M^c_{[\epsilon,\infty)}(\Gamma) := M(\Gamma) - int(M^c_{(0,\epsilon]});$

$N^c_I(\Gamma) := N(\Gamma) \cap M^c_I(\Gamma);$

$\widetilde{N}^c_I(\Gamma) :=$ *preimage of* $N^c_I(\Gamma)$.

Definition 2.10. *Let l be a geodesic in $\mathbb{H}^3$,*

we use τ_l to denote the element in $Isom^+(\mathbb{H}^3)$ which is the 180^o rotation about l.

We end this chapter by the following Proposition which will be used in the latter chapters.

Proposition 2.3. *Let ξ, $\eta \in Isom^+(\mathbb{H}^3)$ with no common fixed point in $\mathbb{S}_\infty$. Then $\exists$ unique geodesic $l = l(\xi, \eta)$ in $\mathbb{H}^3$ such that $\tau_l \xi \tau_l = \xi^{-1}$ and $\tau_l \eta \tau_l = \eta^{-1}$.*

Proof. Firstly, we should consider whether the axis of ξ or η exists or not. If the axis exists, then by the requirment, l must be a geodesic prependicular to the axis. If the axis does not exists, i.e. the number of fixed point is 1, then l must be a geodesic with that fixed point as a end point. So according to the type of ξ and η, l must be a common prependicular geodesic of two geodesics, a geodesic with a fixed end point which is prependicular to a fixed geodesic or a geodesic connecting two fixed end point. In all these cases, l must be uniquely determined. (By Section III.2 [15].) If l is chosen as above, then we can show that τ_l satisfy the above requirment be taking l as a geodesic connecting 0 and ∞ first and checking all cases of ξ and η. □

Remark. $l(\xi, \eta)$ depends continuously on ξ, η.

Chapter 3

Poisson kernel and Conformal densities

In this chapter, we first write down the construction of the Poisson kernel which is used in this paper. Then we study the definition of conformal densities by the use of Poisson kernel. After that, some corresponding results are demostrated as they are important for telling us whether the hypothesis of the theorems used in the latter chapters hold.

3.1 Poisson kernel

In this section, we quoto many results about Poisson kernel which are used in the next section.

Definition 3.1. *We define* $\Pi : \mathbb{H}^n \times \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ *by:*

$$\Pi(z, z', w) = \exp(d(z, w) - d(z', w)).$$

Then by considering the Hyperbolic law of cosines:

$$\cosh a = (\cosh b)(\cosh c) - (\sinh b)(\sinh c)(\cos \alpha),$$

where a, b, c are the sides of a hyperbolic triangle and α is the angle opposite to a. We can extand Π to obtain another continuous function $\overline{\Pi}$.

Definition 3.2. *We define* $\overline{\Pi} : \mathbb{H}^n \times \mathbb{H}^n \times \overline{\mathbb{H}^n} \to \mathbb{R}$ *by:*

$$\overline{\Pi}(z, z', w) = \begin{cases} \Pi(z, z', w) & \text{if } w \in \mathbb{H}^n \\ (\cosh(d(z, z')) - \sinh(d(z, z')) \cos(\angle z' z w))^{-1} & \text{if } w \in \mathbb{S}_\infty \end{cases}$$

Remark. $\overline{\Pi}$ is invariant under isomerty.

Now, we define the Poisson kernel $P := \overline{\Pi}|_{\mathbb{H}^n \times \mathbb{H}^n \times \mathbb{S}_\infty}$. Then P have the following properties:

Lemma 3.1. $P(z, z'', \xi) = P(z, z', \xi) P(z', z'', \xi)$, $\forall z, z', z'' \in \mathbb{H}^n$, $\xi \in \mathbb{S}_\infty$.

Proof. Firstly, we know that this property is satisfied when P is replaced by Π and ξ is in $\mathbb{H}^n$ as exponential function satisfy this property. Then, as $\overline{\Pi}$ is continuous, this property is also hold for P with $\xi \in \mathbb{S}_\infty$. □

Proposition 3.1. $P(z, z', \infty) = \frac{Im(z')}{Im(z)}$, $\forall z, z' \in \mathbb{H}^n$,

where $Im(z)$ denote the last coordiinate of z in the upper half plane.

Proof. Let $z = (x, t), z' = (x', t')$, where $x, x' \in \mathbb{R}^{n-1}$, $t, t' \in (0, \infty)$. Let $w_j = (x, j)$, $w'_j = (x', j)$, $\forall j \in \mathbb{N}$. Then both of w_j and w'_j converge to ∞ as j goes to ∞. As $d(z, w_j) = |\log(\frac{j}{t})|$, $d(z', w'_j) = |\log(\frac{j}{t'})|$ and $\lim\limits_{j\to\infty} d(w_j, w'_j) = 0$. We must have:

$$P(z, z', \infty) = \lim_{j\to\infty} \exp(d(z, w_j) - d(z', w'_j)) = \exp(\log\frac{t'}{t}) = \frac{Im(z')}{Im(z)}.$$

□

Remark. By using isometry, we can find $P(z, z', w)$, $\forall z, z' \in \mathbb{H}^n$, $w \in \mathbb{S}_\infty$.

We will end this section by stating a relation between Possion kernel and conformal expansion factor.

Definition 3.3. *Let $z \in \mathbb{H}^3$, then there is a isometry h that map z to 0 in the Poincaré model. Then the round metric centered at z is the pull-back of the Euclidean metric of the boundary $\mathbb{S}^2$ via h.*

Suppose now we have the round metric centered at z and a isometry γ of $\mathbb{H}^3$. Then as γ acts as a conformal map on $\mathbb{S}_\infty$, we have:

$$d\gamma : T_\xi(\mathbb{S}_\infty) \to T_{\gamma(\xi)}(\mathbb{S}_\infty) \text{ satisfies } |d\gamma(v)| = \lambda_{\gamma,z}(\xi)|v|,\ \forall v \in T_{\gamma(\xi)}(\mathbb{S}_\infty),$$

where $|\cdot|$ denote the length induced by round metric.

Remark. $\lambda_{\gamma,z} : \mathbb{S}_\infty \to (0,\infty)$ is smooth. $\lambda_{\gamma,z}(\xi)$ is called conformal expansion factor of γ at ξ.

From Lemma 3.4.2 [23], we have a relation between Poisson kernel and conformal expansion factor.

Proposition 3.2. *Let $z \in \mathbb{H}^3$, $\gamma \in Isom^+(\mathbb{H}^3)$. Then:*

$$\lambda_{\gamma,z}(\xi) = P(z, \gamma^{-1}(z), \xi) = P(\gamma(z), z, \gamma(\xi)),\ \forall \xi \in \mathbb{S}_\infty.$$

3.2 Conformal densities

In the previous section, we have a discussion on Poisson kernel and conformal expansion factors. In this section, we will continuous our discussion by making use of Poisson kernel to introduce a family of Borel measures on $\mathbb{S}_\infty$ indexed by elements of $\mathbb{H}^n$. This family of Borel measures is called conformal densities.

Definition 3.4. *Let $n \geq 2$, $n \in \mathbb{Z}$, $D \in [0, n-1]$. A D-conformal density for $\mathbb{S}_\infty^{n-1} = \partial\mathbb{H}^n = \mathbb{S}_\infty$ is a family $M = (\mu_z)_{z\in\mathbb{H}^n}$ of finite Borel measures on $\mathbb{S}_\infty$ such that $d\mu_{z'} = P(z, z', \cdot)^D d\mu_z$, $\forall z, z' \in \mathbb{H}^n$.*

Remark. Such D is unique and called degree of M when M is nontrivial.

The following proposition provide us the existence of conformal density and the coincidance of two conformal density which is useful in the proof of the latter results.

Proposition 3.3. *Let $z_0 \in \mathbb{H}^n$, μ be a finite Borel measure on $\mathbb{S}_\infty$, $D \geq 0$. Then there exists unique D-conformal density $M = (\mu_z)$ such that $\mu_{z_0} = \mu$.*

Proof. Uniqueness: As a D-conformal density, the required density must satisfies $\mu_z(E) = \int_E P(z_0, z, \cdot)^D d\mu_{z_0}$, $\forall z \in \mathbb{H}^n$, $E \subset \mathbb{S}_\infty$, measurable. Hence this must be unique.

Existence: We define $\mu_z(E) = \int_E P(z_0, z, \cdot)^D d\mu_{z_0}$, $\forall z \in \mathbb{H}^n$, $E \subset \mathbb{S}_\infty$, measurable. By Lemma 3.1, $\mu_{z'}(E) = \int_E P(z_0, z', \cdot)^D d\mu_{z_0} = \int_E P(z_0, z, \cdot)^D P(z, z', \cdot)^D d\mu_{z_0} = \int_E P(z, z', \cdot)^D d\mu_z$, $\forall z \in \mathbb{H}^n$, $E \subset \mathbb{S}_\infty$, measurable. Therefore there exists such D-conformal density. □

Remark. $supp(\mu_z) = supp(\mu_{z'})$, $\forall z, z' \in \mathbb{H}^n$. We define $supp(M) := supp(\mu_z)$ and $u(z) := \mu_z(\mathbb{S}_\infty)$ when $M = (\mu_z)$ is understood.

Similar as other measures, conformal densities have the following properties:

(i) If $M = (\mu_z)_{z\in\mathbb{H}^n}$, $M' = (\mu'_z)_{z\in\mathbb{H}^n}$ are D-conformal densities, then $M + M' = (\mu_z + \mu'_z)_{z\in\mathbb{H}^n}$ is a D-conformal density.

(ii) If M_i, $i = 1, ..., k$ are D-conformal densities, then $\sum_{i=1}^{k} M_i$ is a D-conformal density.

(iii) If $M = (\mu_z)_{z\in\mathbb{H}^n}$ is a D-conformal density, f is a continuous and nonnegative function, then $N = (\mu'_z)_{z\in\mathbb{H}^n}$ with $d\mu'_z = fd\mu_z$ is a D-conformal density and denoted by $dN = fdM$.

Definition 3.5. *Let A_z be area measure determined by round metric centered at z with $A_z(\mathbb{S}_\infty) = 1$. Then $A = (A_z)_{z\in\mathbb{H}^n}$ is a $(n-1)$-conformal density and is called area density.*

Remark. By Proposition 3.2, area density is really a $(n-1)$-conformal density. Below we introduce the pull-back of a conformal density.

Proposition 3.4. *Let $M = (\mu_z)_{z\in\mathbb{H}^n}$ be a D-conformal density, $\gamma : \mathbb{H}^n \to \mathbb{H}^n$ be isometry,*

Then $(\gamma^*\mu_{\gamma(z)})_{z\in\mathbb{H}^n}$ is also a D-conformal density.

Proof. Let $E \subset \mathbb{S}_\infty$ be measurable. Then we have:

$$\begin{aligned}
\gamma^*\mu_{\gamma(z')}(E) &= \int_E d\gamma^*\mu_{\gamma(z')}(x) \\
&= \int_{\gamma(E)} d\mu_{\gamma(z')}(w) \quad (\text{consider } w = \gamma(x)) \\
&= \int_{\gamma(E)} P(\gamma(z), \gamma(z'), w)^D d\mu_{\gamma(z)}(w) \\
&= \int_{\gamma(E)} P(z, z', \gamma^{-1}(w))^D d\mu_{\gamma(z)}(w) \\
&= \int_E P(z, z', x)^D d\gamma^*\mu_{\gamma(z)}(x)
\end{aligned}$$

□

Remark. The resulting D-conformal density is called the pull-back of M and denoted by $\gamma^* M$.

Proposition 3.5. *Let $f : \mathbb{S}_\infty \to [0, \infty)$ be continuous, $N = (\mu_z)$ be a $(n-1)$-conformal density defined by $dN = fdA$. Define $\overline{u} : \overline{\mathbb{H}^n} \to [0, \infty)$ by:*

$$\overline{u}(z) = \begin{cases} u(z) = \mu_z(\mathbb{S}_\infty) & \text{if } z \in \mathbb{H}^n \\ f(z) & \text{if } z \in \mathbb{S}_\infty \end{cases}$$

Then $\overline{u}$ is continuous.

Proof. Firstly, we have $\overline{u}$ is continuous on $\mathbb{H}^n$ already as P is continuous.

Let $\xi_0 \in \mathbb{S}_\infty$, $(z_i)_{i\in\mathbb{N}} \subset \mathbb{H}^n$ such that z_i converge to ξ_0. Then:

$$\overline{u}(z_i) = \int_{\mathbb{S}_\infty} P(0, z_i, \xi)^{n-1} f(\xi) dA_0(\xi)$$

$$\overline{u}(\xi_0) = f(\xi_0) = \int_{\mathbb{S}_\infty} P(0, z_i, \xi)^{n-1} f(\xi_0) dA_0(\xi)$$

$$\overline{u}(z_i) - \overline{u}(\xi_0) = \int_{\mathbb{S}_\infty} (f(\xi) - f(\xi_0)) P(0, z_i, \xi)^{n-1} dA_0(\xi)$$

Let $\epsilon > 0$ be given. As f is continuous, there exists a neigborhood U of ξ_0 such that:

$$|f(\xi) - f(\xi_0)| < \epsilon,\ \forall \xi \in U \text{ and } \cos\angle\xi_0 0\xi \leq c < 1,\ \forall \xi \in \mathbb{S}_\infty - U,$$

where c is a constant.

Then $\overline{u}(z_i) - \overline{u}(\xi_0) = \int_U (f(\xi) - f(\xi_0)) P(0, z_i, \xi)^{n-1} dA_0(\xi) + \int_{\mathbb{S}_\infty - U} (f(\xi) - f(\xi_0)) P(0, z_i, \xi)^{n-1} dA_0($

Both terms go to 0 as i goes to ∞. ($\because P(0, z_i, \xi) \to 0$ as $i \to \infty$ when $\xi \in \mathbb{S}_\infty - U$ and f is bounded as $\mathbb{S}_\infty$ is compact.) As a result, $\overline{u}$ is continuous. □

Apart from Proposition 3.3, the following proposition give us another condition about whether two conformal densities coincide.

Proposition 3.6. *Let $M = (\mu_z)$ and $M' = (\mu'_z)$ be two $(n-1)$-conformal densities for $\mathbb{H}^n$. Suppose $\mu_z(\mathbb{S}_\infty) = \mu'_z(\mathbb{S}_\infty)$, $\forall z \in \mathbb{H}^n$. Then $M = M'$*

Proof. Consider Poincaré model, then every points of $\mathbb{H}^n$ can be represented uniquely by $t\theta$, where $t \in [0,1)$, $\theta \in \mathbb{S}^{n-1}$. Let f be a continuous and nonnegative function on $\mathbb{S}_\infty$, $N = (\tilde{\mu}_z)$ be a $(n-1)$-conformal density with $d\tilde{\mu}_z = f dA_z$, $\forall z \in \mathbb{H}^n$. Then we have $\int_{\mathbb{S}_\infty} f d\mu_0 = \lim_{t\to 1^-} \int_{\mathbb{S}_\infty} \tilde{\mu}_{t\xi}(\mathbb{S}_\infty) d\mu_0(\xi)$ by Proposition 3.5. By Fubini Theorem, we have:

$$\begin{aligned}
\int_{\mathbb{S}_\infty} \tilde{\mu}_{t\xi}(\mathbb{S}_\infty) d\mu_0(\xi) &= \int_{\mathbb{S}_\infty}\int_{\mathbb{S}_\infty} f(\theta) P(0, t\xi, \theta)^{n-1} dA_0(\theta) d\mu_0(\xi) \\
&= \int_{\mathbb{S}_\infty}\int_{\mathbb{S}_\infty} f(\theta) P(0, t\theta, \xi)^{n-1} d\mu_0(\xi) dA_0(\theta) \\
&= \int_{\mathbb{S}_\infty} \mu_{t\theta}(\mathbb{S}_\infty) f(\theta) dA_0(\theta)
\end{aligned}$$

Similarly, we have $\int_{\mathbb{S}_\infty} \tilde{\mu}_{t\xi}(\mathbb{S}_\infty) d\mu'_0(\xi) = \int_{\mathbb{S}_\infty} \mu'_{t\theta}(\mathbb{S}_\infty) f(\theta) dA_0(\theta)$ and hence $\int_{\mathbb{S}_\infty} f d\mu_0 = \int_{\mathbb{S}_\infty} f d\mu'_0$, for any continuous function f on $\mathbb{S}_\infty$. As a result $\mu_0 = \mu'_0$ and hence $M = M'$, by Proposition 3.3. □

Now, with Proposition 3.3 and 3.6, we know that if $M = (\mu_z)$ and $M' = (\mu'_z)$ are two $(n-1)$-conformal densities and either $\mu_z = \mu'_z$ for some z or $\mu_z(\mathbb{S}_\infty) = \mu'_z(\mathbb{S}_\infty)$ for all z holds, then $M = M'$.

In the rest of this section, we go back to our concerning case and consider only Γ-invariant conformal densities and give a description of the relation bewteen superharmonic functions and conformal densities.

Definition 3.6. *M is called Γ-invariant if $\gamma^* M = M$, $\forall \gamma \in \Gamma$.*

Lemma 3.2. *If M is Γ-invariant, then u is also Γ-invariant.*

Proof.

$$\begin{aligned} u(\gamma(z)) &= \mu_{\gamma(z)}(\mathbb{S}_\infty) \\ &= (\gamma^{-1})^* \mu_z(\mathbb{S}_\infty) \quad (\because M \text{ is } \Gamma\text{-invariant}) \\ &= \mu_z(\mathbb{S}_\infty) \quad (\because \gamma^{-1}(\mathbb{S}_\infty) = \mathbb{S}_\infty) \\ &= u(z) \end{aligned}$$

□

The next two theorems are the main results of this section.

Theorem 3.1. *Let $\Gamma \leq Isom^+(\mathbb{H}^n)$ be nonelementary and discrete. Then every nontrivial Γ-invariant conformal-density has positive degree.*

Proof. Suppose there exists nontrivial Γ-invariant 0-conformal density $M = (\mu_z)_{z \in \mathbb{H}^n}$. Then there exists nontrivial Γ-invariant finite Boreal measure μ on $\mathbb{S}_\infty$. As Γ is nonelementary, there exists a loxodromic element $\gamma \in \Gamma$. (By Lemma 2.3.1 [21].) Let $P, Q \in \mathbb{S}_\infty$ be the fixed points of γ, $\xi \in \mathbb{S}_\infty - \{P, Q\}$. Let U be a neighborhood of ξ such that $\gamma^k(U) \cap U = \emptyset$, $\forall k > 0$. As μ is finite, we must have $\mu(U) = 0$. (Otherwise $\sum_{k=0}^{\infty} \mu(\gamma^k(U)) = \infty$.) Therefore $supp(\mu) \subset \{P, Q\}$. (As ξ

is arbitrary.) As $supp(\mu) \subset \{P, Q\}$, we must have $\Lambda(\Gamma)$ contains at most three points and hence must be exactly two points. ($\because$ two loxodromic elements with exactly one common fixed point will generate a group which is not discrete.(by Theorem 5.1.2 [6].)) Contradiction occur as Γ is nonelementary. $\square$

Theorem 3.2. *Let $\Gamma \leq Isom^+(\mathbb{H}^n)$ be nonelementary and discrete. Suppose every Γ-invariant positive superharmonic function on $\mathbb{H}^n$ is constant. Then all Γ-invariant conformal density is a constant multiply of the area density A.*

Proof. Let $M = (\mu_z)$ be a nontrivial Γ-invariant D-conformal density, where $D \in (0, n-1]$. Consider $u(z) = \mu_z(\mathbb{S}_\infty)$, then $\Delta u = -D(n-D-1)u$ as P satisfy the same relation. (By Theorem 5.13 [23].) Therefore u is superharmonic and hence $u(z) = C$, $\forall z \in \mathbb{H}^n$, where C is a constant. Therefore $\Delta u = -D(n-D-1)u = 0$ and hence $D = n-1$ as M is nontrivial. By Proposition 3.6, we must have $M = CA$. $\square$

Chapter 4

Patterson construction and decomposition

In this chapter, we continue the study of conformal density of last chapter and make use of the knowledge of Poincaré series to have some constructions which will lead to some theorems that are similar to our main result. Most of the basic idea can be acquire from chapter 3, [23] and here we will illustrate the work of Shalen and Culler[13] on the extension of these idea.

4.1 Patterson construction

In this section, we will introduce Poincaré series together with its properties and some extensions by following Shalen and Culler's method, where these kind of method is first introduced by Patterson.

Definition 4.1. *$W \subset \mathbb{H}^n$ is called uniformly discrete if there exists $\epsilon > 0$ such that $d(z, w) > \epsilon$, $\forall z, w \in W$.*

Remark. ϵ is called modulus of discreteness for W. Γw is uniformly discrete $\forall w \in \mathbb{H}^n$ if $\Gamma \leq Isom^+(\mathbb{H}^n)$ is discrete.

Definition 4.2. *Let $W \subset \mathbb{H}^n$ be uniformly discrete. Then $\overline{W} = W \cup \Lambda_W \subset \overline{\mathbb{H}^n}$ with $\Lambda_W \subset \mathbb{S}_\infty$. Λ_W is called the limit set of W.*

By Theorem 5.3.9 [6], we have the following lemma:

Lemma 4.1. *Let Γ be a Kleinian group, $w \in \mathbb{H}^3$. Let $W = \Gamma w$. Then $\Lambda = \overline{W} \cap \mathbb{S}_\infty = \Lambda(\Gamma)$.*

We now introduce what Poincaré series is:

Definition 4.3. *Let $z \in \mathbb{H}^n$, $s > 0$, $W \subset \mathbb{H}^n$ be uniformly discrete. The Poincaré series for W is:*

$$\Sigma(z, s) := \sum_{w \in W} e^{-sd(z,w)}.$$

From now on, we assume $\Sigma(z, s)$ is the Poincaré series for a given uniformly discrete set W unless other specify. By the properties of exponential function, we have:

Proposition 4.1. $\forall z, z' \in \mathbb{H}^n$, $s' \geq s \geq 0$.

$\Sigma(z, s) \leq \Sigma(z, s')$;

$\Sigma(z', s) \leq e^{sd(z,z')}\Sigma(z, s)$.

By Theorem 1.6.1 [23], we have:

Lemma 4.2. *There exists unique $D \in [0, n-1]$ such that $\forall z \in \mathbb{H}^n$, we have:*

(i) $\Sigma(z, s) < \infty$, $\forall s > D$;

(ii) $\Sigma(z, s) = \infty$, $\forall 0 \leq s < D$.

Remark. Such D is called critical exponent of W and it is independent of z by Proposition 4.1.

For our usage, we want to modify Poincaré series by using adjustment functions.

Definition 4.4. *$f : [0, \infty) \to \mathbb{R}$ is an adjustment functions if: (i) f is C^1, (ii) $f(0) = 0$, (iii) $0 \leq f'(t) \leq 1$, $\forall t > 0$ and (iv) $\lim_{t\to\infty} f'(t) = 1$.*

Then we defind $\Sigma(f, z, s) := \sum_{w \in W} e^{-sf(d(z,w))}$, $\forall f : [0, \infty) \to \mathbb{R}$ adjustment function.

Here we write down the construction of a suitable adjustment function now, so that we can use the modified Poincaré series to form a Borel measure and hence a conformal density.

Lemma 4.3. *Let D be the critical exponent of the given discrete set W. Then there exists an adjustment function f such that $\forall z \in \mathbb{H}^n$, we have:*

$$\Sigma(f, z, s) < \infty, \forall s > D$$

$$\Sigma(f, z, s) = \infty, \forall 0 \leq s \leq D.$$

Proof. Firstly, we observe that $\Sigma(f, z, s) \geq \Sigma(z, s)$, $\forall s$ as $f(t) \leq t$. Therefore it is true when $s < D$. For $s > D$, as $\lim_{t \to \infty} f'(t) = 1$, there exists $k < \frac{D}{s} < 1$, $C \in \mathbb{R}$ such that $\Sigma(f, z, s) \leq e^{Cs}\Sigma(z, sk) < \infty$. Therefore we only need to consider the case when $s = D$.

Case 1: $D = 0$

We define f as identity map, then $\Sigma(f, z, 0) = \infty$ as $|W| = \infty$.

Case 2: $D > 0$

As $\Sigma(f, z, D) = \infty$ for some $z \in \mathbb{H}^n$ lead to the conclusion $\Sigma(f, z, D) = \infty$, $\forall z \in \mathbb{H}^n$, we may fix a $z \in \mathbb{H}^n$ first.

Define $\theta_m = \frac{m}{m+1}$, $\forall m \in \mathbb{N}$. Let $s_m = \theta_m D$, $\forall m \in \mathbb{N}$. As $0 < s_m < D$, we have $\Sigma(z, s_m) = \infty$. Therefore there exists $X_m \subset W$ with $|X_m| < \infty$ such that $\sum_{w \in X_m} e^{-s_m d(z,w)} \geq m$, $\forall m$. Define $R_1 = \max_{w \in X_1}\{d(z, w)\}$, $R_m = \max_{w \in X_m}\{d(z, w), R_{m-1} + 1\}$, $\forall m > 1$. Then we have $R_m \to \infty$ as $m \to \infty$ and $d(z, w) \leq R_m$, $\forall z \in X_m$. Let $\beta : [0, \infty) \to [0, \infty)$ be continuous, monotone increasing function such that:

$$\beta(R_m) = \theta_m, \forall m \in \mathbb{N} \text{ and } \beta(t) \leq 1, \forall t \geq 0$$

For example, we may take β to be the piecewise linear function that satisfies the conditions. Define $f(t) = \int_0^t \beta(x)dx$. Then $f(t) \leq \theta_m t$, $\forall 0 \leq t \leq R_m$. Therefore $f(d(w,z))D \leq \theta_m Dd(w,z) = s_m d(w,z)$, $\forall w \in X_m$, $d(w,z) < R_m$. Therefore $\Sigma(f,z,D) \geq \sum_{w\in X_m} e^{-Df(d(w,z))} \geq \sum_{w\in X_m} e^{-s_m d(w,z)} \geq m$. As $|W| = \infty$, we have $\Sigma(f,z,D) = \infty$. $\square$

From now on, we always assume f is an adjustment function consructed in Lemma 4.3 unless other specify.

Let B be a countable collection of subsets of W, $W \in B$. Then $\forall V \in B$, we can defind a Boreal measure on $\overline{\mathbb{H}^n}$ by:

$$\mu_{V,z,s} := \sum_{w\in V} e^{-sf(d(w,z))}\delta_w$$

Then the total mass $\leq \Sigma(f,z,s)$ with equality hold when $V = W$. With these setting, we have the following lemma.

Lemma 4.4. *Let $z_0 \in \mathbb{H}^n$ be given. Then there exists $(s_j)_{j\in\mathbb{N}}$ such that:*

(i) $s_j > D$, $\forall j$ and $\lim_{j\to\infty} s_j = D$;

(ii) $\forall V \in B$, $\Sigma(f,z_0,s_j)^{-1}\mu_{V,z_0,s_j}$ converge weakly to a Borel measure on $\overline{\mathbb{H}^n}$.

Proof. Let $(t_i)_{i\in\mathbb{N}}$ be a sequence satisfy (i). As total mass of $\Sigma(f,z_0,t_i)^{-1}\mu_{V,z_0,t_i} \leq 1$, there exists a subsequence of $\Sigma(f,z_0,t_i)^{-1}\mu_{V,z_0,t_i}$ that converge weakly. As B is countable, we can apply diagonalization to finish the proof. $\square$

For simplicity, assume $(s_j)_{j\in\mathbb{N}}$ satisfies the result of Lemma 4.4, then we let $C_j := \Sigma(f, z_0, s_j)$, μ_{V,z_0} to be the weak limit, where $z_0 \in \mathbb{H}^n$ is assumed to be given. Then we have $C_j \to \infty$ as $j \to \infty$. ($\because s_j \to D$ as $j \to \infty$.)

Lemma 4.5. *$\forall V \in B$, $supp(\mu_{V,z_0}) \subset \mathbb{S}_\infty$.*

Proof. Let $V \in B$, $K \subset \mathbb{H}^n$ be compact, then $\mu_{V,z_0,s_j}(K) \leq |V \cap K|$, $\forall j \in \mathbb{N}$. Therefore $\frac{1}{C_j}\mu_{V,z_0,s_j}(K) \leq \frac{1}{C_j}|V \cap K| \to 0$ as $j \to \infty$. Therefore $supp(\mu_{V,z_0}) \subset \mathbb{S}_\infty$. □

We end this section with the following main theorem of this section.

Theorem 4.1. *Let $W \subset \mathbb{H}^n$ be uniformly discrete, $|W| = \infty$. Let B be a countable collection of subsets of $W, W \in B$. Then $\exists\ D \in [0, n-1]$, $(M_V)_{V\in B}$, M_V is D-conformal density of $\mathbb{H}^n$, $\forall V \in B$ such that:*

(i) M_W is nontrivial.

(ii) Let $(V_i)_{1\leq i\leq k}$ be disjoint, $V = \bigcup_{i=1}^{k} V_i$. Then $M_V = \sum_{i=1}^{k} M_{V_i}$.

(iii) Let $V \in B$, $\gamma : \mathbb{H}^n \to \mathbb{H}^n$ be isometry such that $\gamma(V) \in B$. Then $\gamma^*(M_{\gamma(V)}) = M_V$.

(iv) Let $V \in B$. Then $supp(M_V) \subset$ limit set of V.

Remark. As a consequence of (iv), we have M_V is trivial if V is finite.

Proof. By Lemma 4.4 and 4.5, consider the Poincaré model, there exists $D \in [0, n-1]$, $\mu_{V,0}$ Borel measure on $\mathbb{S}_\infty$, $\forall V \in B$. By Proposition 3.3, there exists unique D-conformal density $M_V = (\mu'_{V,z})$ such that $\mu'_{V,0} = \mu_{V,0}$, $\forall V \in B$. Now we only need to show (i)-(iv) are satisfied.

(i) By our constuction of $\mu'_{W,0}$ in Lemma 4.4, we have total mass of $\mu'_{W,0} = 1$.

(ii) Let $V = \bigcup_{i=1}^{k} V_i$, $V, V_i \in B$. Then by our construction, we must have $\mu'_{V,0} = \sum_{i=1}^{k} \mu'_{V_i,0}$. Then by Proposition 3.3, we also have $\mu'_{V,z} = \sum_{i=1}^{k} \mu'_{V_i,z}$, $\forall z \in \mathbb{S}_\infty$. Therefore $M_V = \sum_{i=1}^{k} M_{V_i}$.

(iii) Let $V \in B$, $\gamma \in Isom^+(\mathbb{H}^n)$ such that $\gamma(V) \in B$. Let $s > D$, then $\gamma^*(\frac{1}{C_j}\mu'_{\gamma(V),\gamma(0),s}) = \frac{1}{C_j}\sum_{w \in V} e^{-sf(d(\gamma(w),\gamma(0)))}\gamma^*(\delta_{\gamma(w)}) = \mu'_{V,0,s}$ as γ are isometry and $\gamma(V) \subset W$. Then it is true with similar reason of (ii).

(iv) By our construction, $supp(\mu'_{V,z}) \subset V \cup \Lambda_V$. As $supp(\mu'_{V,z}) \subset \mathbb{S}_\infty$, we must have $supp(\mu'_{V,z}) \subset \Lambda_V$. $\square$

By Theorem 4.1, we have:

Corollary 4.1. *Let $\Gamma \leq Isom^+(\mathbb{H}^n)$ be nonelementary and discrete. Then there exists Γ-invariant nontrivial conformal density M for $\mathbb{H}^n$ with $supp(M) = \Lambda(\Gamma)$.*

4.2 Patterson decomposition

In this section, we study the usage of the result obtained in the last section on proving some theorem that is similar to our main result.

Lemma 4.6. *Let Γ be a Kleinian group which is free on a generating set G, $2 \leq |G| < \infty$. Let $\Psi = G \cup G^{-1} \subset \Gamma$, $z_0 \in \mathbb{H}^3$. Then there exists $D \in [0,2]$, $M = (\mu_z)$ a Γ-invariant D-conformal density for $\mathbb{H}^3$, $(v_\psi)_{\psi \in \Psi}$ a family of Borel measure on $\mathbb{S}_\infty$ such that:*

(i) $\mu_{z_0}(\mathbb{S}_\infty) = 1$;

(ii) $\mu_{z_0} = \sum_{\psi \in \Psi} v_\psi$;

(iii) $\int_{\mathbb{S}_\infty} (\lambda_{\psi, z_0})^D dv_{\psi^{-1}} = 1 - \int_{\mathbb{S}_\infty} dv_\psi$.

Suppose $G = \{\xi, \eta\}$ and $z_0 \in l(\xi, \eta)$, then we have:

(iv) $\int_{\mathbb{S}_\infty} dv_{\xi^{-1}} = \int_{\mathbb{S}_\infty} dv_\xi$ *and* $\int_{\mathbb{S}_\infty} dv_{\eta^{-1}} = \int_{\mathbb{S}_\infty} dv_\eta$.

Proof. Let $1 \neq \gamma \in \Gamma$, then $\gamma = \psi_1 \cdots \psi_k$, $k \geq 0$, $\psi_i \in \Psi$, $\psi_{i+1} \neq \psi_i^{-1}$. This representation is unique and we call ψ_1 the initial letter of γ. Let $J_\psi := \{\gamma \in \Gamma : \text{initial letter of } \gamma = \psi\}$, $\psi \in \Psi$. Then $\Gamma = \coprod_{\psi \in \Psi} J_\psi \coprod \{1\}$. Let $W = \Gamma z_0 = \coprod_{\psi \in \Psi} V_\psi \coprod \{z_0\}$, where $V_\psi = J_\psi z_0$. Let $B = \{\coprod_{\psi \in \Psi'} V_\psi, \coprod_{\psi \in \Psi'} V_\psi \coprod \{z_0\} : \Psi' \subset \Psi\}$. Then $|B| < \infty$ as $|G| < \infty$. By Theorem 4.1, there exists $D \in [0,2]$, $(M_V)_{V \in B}$ satisfy the result of 4.1. Let $(\mu_z) = M = M_W$, $v_\psi = \mu_{V_\psi, z_0}$, where $M_{V_\psi} = (\mu_{V_\psi, z})$. Then M is Γ-invariant D-conformal and we only need to check (i)-(iv).

(i) By multiplying a constant, we may assume $\mu_{z_0}(\mathbb{S}_\infty) = 1$.

(ii) $\mu_{z_0} = \mu_{W, z_0} = \mu_{\{z_0\}, z_0} + \sum_{\psi \in \Psi} \mu_{V_\psi, z_0} = \sum_{\psi \in \Psi} \mu_{V_\psi, z_0}$. (By Theorem 4.1 (ii).)

(iii) We have $\psi J_{\psi^{-1}} = \Gamma - J_\psi$, $\psi V_{\psi^{-1}} = W - V_\psi$. Therefore $W - V_\psi = \coprod_{\psi' \neq \psi} V_{\psi'} \coprod \{z_0\} \in B$. By Theorem 4.1 (iii), we have:

$$M_{V_{\psi^{-1}}} = \psi^*(M_{\psi V_{\psi^{-1}}}) = \psi^*(M_{W-V_\psi}) = \psi^*(M - M_{V_\psi}).$$

As a result, we have:

$$\mu_{V_{\psi^{-1}},\psi^{-1}(z_0)} = \psi^*(\mu_{z_0} - \mu_{V_\psi,z_0}) = \psi^*(\mu_{z_0} - v_\psi).$$

As $d\mu_{V_{\psi^{-1}},\psi^{-1}(z_0)} = P(z_0,\psi^{-1}(z_0),\cdot)^D dv_{\psi^{-1}} = \lambda^D_{\psi,z_0} dv_{\psi^{-1}}$, we have:

$$\int_{\mathbb{S}_\infty} \lambda^D_{\psi,z_0} dv_{\psi^{-1}} = \int_{\mathbb{S}_\infty} d\psi^*(\mu_{z_0} - v_\psi) = 1 - \int_{\mathbb{S}_\infty} dv_\psi.$$

(iv) Let $G = \{\xi,\eta\}$, $z_0 \in l(\xi,\eta)$, $\tau = \tau_l$. Then as $\tau\xi\tau = \xi^{-1}$ and $\tau\eta\tau = \eta^{-1}$, we have $\tau\Gamma\tau = \Gamma$, $\tau J_\xi \tau = J_{\xi^{-1}}$ and $\tau J_\eta \tau = J_{\eta^{-1}}$. Hence we have $V_{\xi^{-1}} = \tau V_\xi$. Therefore $\tau^* M_{V_\xi} = M_{V_{\xi^{-1}}}$, $\tau^* v_\xi = v_{\xi^{-1}}$, $(\because \tau(z_0) = z_0.)$ So we have $\int_{\mathbb{S}_\infty} dv_{\xi^{-1}} = \int_{\mathbb{S}_\infty} dv_\xi$. Similarly, $\int_{\mathbb{S}_\infty} dv_{\eta^{-1}} = \int_{\mathbb{S}_\infty} dv_\eta$ is also true. □

Lemma 4.7. *Let (X,B) be a measurable space. Let μ,μ_0 be measure of (X,B) with $\mu(X)$, $\mu_0(X) < \infty$ and $0 \le \mu_0 \le \mu$. Let $C \in B$ such that $\mu(C) \ge \mu_0(X)$. Let $f : X \to [0,\infty)$ be measurable such that $inf f(C) \ge sup f(X - C)$. Then $\int_X f d\mu_0 \le \int_C f d\mu$.*

Proof. Define $\mu_1 = \mu - \mu_0$. Then μ_1 is a measure of (X,B), $\mu_1 \ge 0$ and we have:

$$\mu_0(X - C) = \mu_0(X) - \mu_0(C) \le \mu(C) - \mu_0(C) = \mu_1(C).$$

$$\int_X f d\mu_0 = \int_C f d\mu_0 + \int_{X-C} f d\mu_0$$
$$\leq \int_C f d\mu_0 + \sup_{x \in X-C}(f)\mu_0(X-C)$$
$$\leq \int_C f d\mu_0 + \inf_{x \in C}(f)\mu_1(C)$$
$$\leq \int_C f d\mu$$

□

In the rest of this section, we prove three important results of this section following Shalen and Culler's method.

Lemma 4.8. *Let $a, b \in \mathbb{R}$ such that $0 \leq a \leq \frac{1}{2}$ and $0 \leq b \leq 1$. Let $\gamma \in Isom^+(\mathbb{H}^3)$, $z \in \mathbb{H}^3$. Suppose v is a measure on $\mathbb{S}_\infty$ such that:*

(i) $v \leq A_z$;

(ii) $v(\mathbb{S}_\infty) \leq a$;

(iii) $\int_{\mathbb{S}_\infty} \lambda^2_{\gamma,z} dv \geq b$.

Then $d(z, \gamma(z)) \geq \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}$.

Proof. Let $h = d(z, \gamma(z))$, $c = coshh$, $s = sinhh$. Then $\lambda_{\gamma,z}(\xi) = (c - s\cos\angle\gamma^{-1}(z)z\xi)^{-1}$. Let $\phi_0 = \arccos(1-2a)$, $C = \{\xi \in \mathbb{S}_\infty : \angle\gamma^{-1}(z)z\xi \geq \phi_0\}$. Then we have:

$$A_z(C) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\phi_0} \sin\phi d\phi d\theta = a.$$

By Lemma 4.7, we will have:

$$b \leq \int_{\mathbb{S}_\infty} \lambda^2_{\gamma,z} dv \leq \frac{a}{(c-s)(c-s+2as)}.$$

Therefore $d(z, \gamma(z)) \geq \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}$.

□

Theorem 4.2. *Let Γ be a Kleinian group which is free on a generating set G such that $2 \leq k < \infty$, where $k = |G|$. Suppose $M = C_M A$, where C_M is a constant, $\forall\Gamma$-invariant conformal density M. Then*

$$\max_{\xi \in G} d(z, \xi(z)) \geq \tfrac{1}{2} \log\left((k-1)(2k-1)\right),\ \forall z \in \mathbb{H}^3$$

Proof. By Lemma 4.6, there exists $D \in [0,2]$, $M = (\mu_z)$, $(v_\psi)_{\psi \in \Psi}$ satisfy the result of 4.6. Here we take $\Psi = G \coprod G^{-1}$. As $M = C_M A$, we have $D = 2$. By Lemma 4.6(i), $\mu_{z_0}(\mathbb{S}_\infty) = 1 \Rightarrow C_M = 1$ and $M = A$. By Lemma 4.6(ii), $1 = \sum\limits_{\psi \in \Psi} v_\psi(\mathbb{S}_\infty) = \sum\limits_{\xi \in G} (v_\xi(\mathbb{S}_\infty) + v_{\xi^{-1}}(\mathbb{S}_\infty))$. Therefore there exists $\xi_0 \in G$ such that $v_{\xi_0}(\mathbb{S}_\infty) + v_{\xi_0^{-1}}(\mathbb{S}_\infty) \leq \frac{1}{k}$. Hence there exists $\psi_0 \in \{\xi_0, \xi_0^{-1}\}$ such that $v_{\psi_0}(\mathbb{S}_\infty) \leq \frac{1}{2k}$. Then we can apply Lemma 4.8 with $z = z_0$, $\gamma = \psi_0$, $a = \frac{1}{2k}$, $b = 1 - \frac{1}{k}$. As a result $d(z, \psi_0(z)) \geq \frac{1}{2} \log\left((k-1)(2k-1)\right)$. Therefore we have:

$$\max_{\xi \in G} d(z, \xi(z)) \geq \tfrac{1}{2} \log\left((k-1)(2k-1)\right),\ \forall z \in \mathbb{H}^3.$$

□

With similar method used in Theorem 4.2, we also have:

Theorem 4.3. *Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$ be loxodromic. Suppose $\Gamma = \langle \xi, \eta \rangle$ is discrete and free on ξ and η. Suppose $M = C_M A$, where C_M is a constant, $\forall\Gamma$-*

invariant conformal density M. Then $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$.

Proof. Let $l = l(\xi,\eta)$. By Proposition 2.1, we know that we only have to show $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, $\forall z \in l$. Now let $z \in l$. By Lemma 4.6(ii) and (iv), we have:

$$1 = 2v_\xi(\mathbb{S}_\infty) + 2v_\eta(\mathbb{S}_\infty).$$

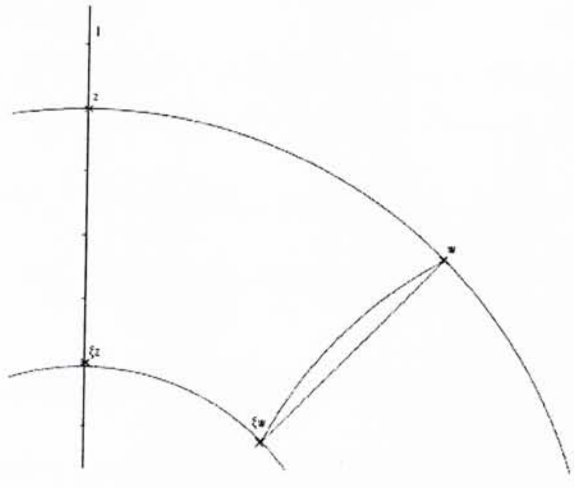
So, we may assume $v_\xi(\mathbb{S}_\infty) \leq \frac{1}{4}$. By Lemma 4.6 (iii) and (iv), we have:

$$\int_{\mathbb{S}_\infty} \lambda^2_{\xi^{-1},z} dv_\xi = 1 - v_\xi(\mathbb{S}_\infty) \geq \tfrac{3}{4}.$$

Then we can apply Lemma 4.8 with $v = v_\xi$, $\gamma = \xi^{-1}$, $a = \frac{1}{4}$, $b = \frac{3}{4}$. As a result $d(z,\xi(z)) \geq \log 3$. Therefore we have:

$$\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3,\ \forall z \in \mathbb{H}^3.$$

□



Chapter 5

Bonahon surfaces and Grided surfaces

In this chapter, we quoto many results of Bonahon [8], Canary [9], Chuckrow [12] and Thurston [27] about Bonahon surfaces and grided surface. With these results, Shalen and Culler provide us some conditions to ensure the hypothesis of Theorem 3.2 hold by stating Proposition 5.6. Theorem 5.1 is proved for more complicated case when we cannot apply Theorem 5.6 directly.

5.1 Bonahon surfaces

In this section, we write down some definitions about Bonahon surface in a given manifold. The features of a Bonahon surface provide some information about the corresponding manifold.

Let Σ be a closed 2-manifold, M be a manifold. Let u be a generator of $H_2(\Sigma;\mathbb{Z}/2)$. Let $f:\Sigma\to M$. Then we have the following definitions:

Definition 5.1. *f is null-homologous if $f_*(u)\in H_2(M;\mathbb{Z}/2)$ is trivial.*

Definition 5.2. *Let $f:\Sigma\to M$, $p\in M-f(\Sigma)$. Then P is called strictly enclosed by f if $f_*(u)\in H_2(M-\{p\};\mathbb{Z}/2)$ is nontrivial.*

p is enclosed by f if $p\in f(\Sigma)$ or p is strictly enclosed by f.

$K\subset M$ compact is enclosed by f if p is enclosed by f, $\forall p\in K$.

By Section 8.12 [27], and Theorem 9.1 [9], we have:

Proposition 5.1. *Let Γ be a Kleinian group without parabolics such that $M(\Gamma)=N(\Gamma)$. Let Σ_j be closed 2-manifold $\forall j\in\mathbb{N}$. Let $f_j:\Sigma_j\to M$ such that $\forall K\subset M(\Gamma)$ compact, K is enclosed by some f_j. Suppose $\exists V>0$ such that $Vol(nbhd_1(f_j(\Sigma_j)))<V$, $\forall j\in\mathbb{N}$. Then every positive superharmonic function on M is constant.*

Then we introduce what virtual triangulation is. Let Δ^k be a k-simplex, X be a topological space.

Definition 5.3. *$\phi : \Delta^k \to X$ is a virtual k-simplex in X if ϕ is injective on each face of Δ^k.*

ϕ, ϕ' define same virtual simplex if $\exists$ isomorphism I between them such that $\phi \circ I = \phi'$.

$\phi' : \Delta' \to X$ is called a face of $\phi : \Delta \to X$ if $\exists$ isomorphism $I : \Delta' \to \widetilde{\Delta}$ a face of Δ such that $\phi \circ I = \phi'$.

$Int(\phi(\Delta)) := \phi(Int(\Delta))$, dimension of $\phi(\Delta) :=$ dimension of Δ.

Then we have:

Definition 5.4. *A virtual triangulation of Σ is a collection Φ of virtual simplices such that:*

(i) Φ contains the faces of each simplex in Φ;

(ii) Σ is the disjoint union of the interiors of the virtual simplices in Φ.

We use the following convention:

$skel_i(\Sigma) :=$ union of the interior of all virtual simplices of dimension $\leq i$.

virtual 0-simplex is called vertices.

A 1-simplex ϕ is called degenerate if ϕ is a trivial loop.

A virtual triangulation is nondegenerate if it has no degenerate 1-simplex.

As if there is a homeomorphism between a simplex and hyperbolic space, it will induce a hyperbolic structure on the simplex. Therefore, we have something called Piecewise hyperbolic structure.

Definition 5.5. *A piecewise hyperbolic structure on a virtual triangulated surface Σ is a family of hyperbolic structure on Σ such that any two induced linear metrics agree.*

Then we denote $\theta_w(\phi)$ to be the angle at a vertex w induced by a virtual 2-simplex ϕ. Let $x \in \Sigma$, we have the following definitions:

Definition 5.6. $a_x := \begin{cases} \sum\limits_{\phi(w)=x} \theta_w(\phi) & \text{if } x \text{ is a vertex} \\ 2\pi & \text{if } x \in int\Sigma,\ x \text{ not a vertex} \\ \pi & \text{if } x \in \partial\Sigma,\ x \text{ not a vertex} \end{cases}$

$excess(x) := \begin{cases} a_x - 2\pi & \text{if } x \in int\Sigma \\ a_x - \pi & \text{if } x \in \partial\Sigma \end{cases}$

x is a singular point if $excess(x) \neq 0$.

x is a corner if $excess(x) < 0$ and $x \in \partial\Sigma$.

$sing\Sigma := \{x \in \Sigma : x \text{ is singular point } \}$.

Then we can define ultrahyperbolic surface.

Definition 5.7. *A ultrahyperbolic surface is a piecewise hyperbolic surface with $excess(x) \geq 0,\ \forall x \in int\Sigma$.*

Let Σ be a ultrahyperbolic surface, $p \in \Sigma$. Then we have the following definitions:

$short(p) := \inf_{1 \neq \gamma \text{ PG loop based at } p} length(\gamma)$, where PG stand for piecewise geodesic.

$\Sigma_I := \{p \in \Sigma : short(p) \in I\}$, $\forall I \subset (0, \infty]$.

$\Sigma_{(0,\epsilon]} :=$ ϵ-thin part of Σ.

$length(\gamma)$ modulo $\Sigma_{(0,\epsilon]} := length(\gamma^{-1}(\Sigma_{(\epsilon,\infty)}))$.

$d(x, y)$ modulo $\Sigma_{(0,\epsilon]} := \inf_{\gamma \text{ PG connecting } x,y} length(\gamma)$ modulo $\Sigma_{(0,\epsilon]}$.

diameter of Σ modulo $\Sigma_{(0,\epsilon]} := \sup_{x,y \in \Sigma} d(x, y)$ modulo $\Sigma_{(0,\epsilon]}$.

Then by Lemma 1.10 [8], we have:

Proposition 5.2. *Let $g \in \mathbb{N}$, $\epsilon > 0$. Then there exists $D(g, \epsilon) \in \mathbb{R}$ such that diameter of Σ modulo $\Sigma_{(0,\epsilon]} < D(g, \epsilon)$, $\forall$ closed ultrahyperbolic surface Σ of genus g.*

Definition 5.8. *Let $M(\Gamma)$ be a complete hyperbolic manifold. Let Σ be a virtually triangulated surface. $f : \Sigma \to M(\Gamma)$ is hyperbolically simplicial if $\forall \phi : \Delta \to \Sigma$ virtual simplex, $f_\phi = f \circ \phi$ admits a continuous lift $\widetilde{f_\phi}$ such that $\widetilde{f_\phi}(\Delta)$ is a hyperbolic simplex and either (i) $dim \widetilde{f_\phi}(\Delta) < dim \Delta$ or (ii) $\widetilde{f_\phi}$ is a homeomorphism.*

Remark. f is called nondegenerate if (ii) holds $\forall$ virtual simplex. We denote (Σ, f) to be the surface Σ with induced structure from f and is called hyperbolically simplicial surface in M.

Below we will write down the definition of Bonahon surface.

Definition 5.9. *Let (Σ, f) be a nondegenerate hyperbolically simplicial surface in M. (Σ, f) is called a Bonahon surface if there exists closed 1-manifold $S \subset \Sigma$ such that $skel_0(\Sigma) \subset S \subset skel_1(\Sigma)$. (Σ, f) is called ϵ-incompressible if $f \circ \gamma$ is homotopically nontrivial in M, $\forall$ PG closed curve $\gamma : \mathbb{S}^1 \to \Sigma$, γ homotopically nontrivial with $length(\gamma) < \epsilon$.*

By Lemma 1.8 [8], we have:

Proposition 5.3. *Let (Σ, f) be a Bonahon surface in a complete hyperbolic manifold M. Then the PH structure induced by f is ultrahyperbolic.*

By lemma 8.2 [9], we have:

Proposition 5.4. *Let $g \in \mathbb{N}$, $\epsilon > 0$. Then there exists $V(g, \epsilon) \in \mathbb{R}$ such that $Vol(nbhd_1(f(\Sigma))) \leq V(g, \epsilon)$, $\forall \epsilon$-incompressible Bonahon surface (Σ, f) of genus g in a complete hyperbolic 3-manifold M.*

By above propositions, we can prove the following proposition.

Proposition 5.5. *Let Γ be a Kleinian group without parabolics and $M(\Gamma) = N(\Gamma)$. Suppose $\exists g_1, ...g_k \in \mathbb{N}$, $\epsilon > 0$ such that $\forall K \subset M(\Gamma)$, K is enclosed by one of $(\Sigma_1, f_1), ..., (\Sigma_k, f_k)$, where (Σ_i, f_i) is ϵ-incompressible Bonahon surface with genus $< g_i$. Then every positive superharmonic function on $M(\Gamma)$ is constant.*

Proof. By Proposition 5.4, we have $Vol(nbhd_1(f_i(\Sigma_i))) \leq V(g_i', \epsilon)$, where g_i' is the genus of Σ_i. Let $V = \max_{1 \leq i \leq k}\{V(g_i', \epsilon)\}$. Then we have $vol(nbhd_1(f_i(\Sigma_i))) \leq V$,

$\forall 1 \leq i \leq k$. Therefore we can get the result by applying Proposition 5.1. $\square$

In fact by Theorem 8.1 and Theorem 9.1 [9], we have the following result which will be used to prove our main result.

Proposition 5.6. *Let Γ be a Kleinian group without parabolics and $M(\Gamma) = N(\Gamma)$. Suppose Γ is topologically tame. Then every positive superharmonic function on $M(\Gamma)$ is constant.*

Proposition 5.7. *Let M be a 3-manifold with an involution T such that every connected component of $FixT$ is homeomorphic to $\mathbb{R}$. Let (Σ, f) be a null-homologus hyperbolically simplicial surface in M. Let T_Σ be an orientation-preserving involution of Σ such that $f \circ T_\Sigma = T \circ f$. Suppose $\exists$ component L of $FixT$ such that there are exactly two fixed points v_+ and v_- of T_Σ with $f(v_+)$, $f(v_-) \in L$. Let s be the compact arc in L bounded by $f(v_+)$ and $f(v_-)$. Then s is enclosed by (Σ, f).*

Proof. Let $p \in s \cap (M - f(\Sigma))$. As (Σ, f) is null-homologus, we have $|f^{-1}(L)| = 2k$, where $k \in \mathbb{Z}$. Let L_+ and L_- be the components of $L - \{p\}$ containing $f(v_+)$ and $f(v_-)$ respectively. ($\because$ L is homeomorphic to $\mathbb{R}$.) Note that we have $f^{-1}(L_+)$ is invarant under T_Σ as $f \circ T_\Sigma(f^{-1}(L_+)) = T \circ f(f^{-1}(L_+)) = L_+$ and $f^{-1}(L_+) = f^{-1}(v_+)$ only. Therefore $|f^{-1}(L_+)|$ is odd and so as $|f^{-1}(L_-)|$. As a result p is strictly enclosed by f. Hence s is enclosed by (Σ, f). $\square$

5.2 Grided surfaces

In this section, we want to study some theorem related to grided surfaces which is helpful for us to solve the most difficult case in our main theorem.

Definition 5.10. *Let $\epsilon > 0$, $M(\Gamma)$ be a complete hyperbolic manifold. Let (Σ, f) be a nondegenerate hyperbolically simplicial surface. Let $S \subset \Sigma$ be a closed 1-manifold. (Σ, f, S) is called an ϵ-girded surface in M if:*

(i) $skel_0(\Sigma) \subset S \subset skel_1(\Sigma)$;

(ii) Each component of $\Sigma - S$ is homeomorphic to a sphere with 3 punctures;

(iii) $f|_S$ is geodesic, length$(f|_C) < \frac{\epsilon}{3}$, $\forall$ component C of S;

(iv) $f|_R$ induces $f_* : \pi_1(R) \to \pi_1(M)$ injective, $\forall$ component R of $\Sigma - S$.

Remark. From the definition, we also have $f|_C$ are reparametrization of closed geodesics and said to be carried by (Σ, f, S). S has $3g - 3$ components, where g is the genus of Σ.

For a girded surface, we have the following propositions:

Proposition 5.8. *Let ϵ be a Margulis number for M. Let (Σ, f, S) be an $\frac{\epsilon}{3}$-girded surface in M. Then (Σ, f) is $\frac{\epsilon}{3}$-incompressible.*

Proposition 5.9. *Let M be a hyperbolic 3-manifold and let ϵ be a Margulis* *number for M. Let (Σ, f, S) be an $\frac{\epsilon}{3}$-girded surface of total genus g in M. Let $\gamma_1, ..., \gamma_k$ be the distinct geodesics carried by (Σ, f, S). Let T_i be the component of $M_{(0,\epsilon]}$ containing γ_i. Then $f(\Sigma) \subset nbhd_D(T_1 \cup ... \cup T_k)$, where $D = D(g, \frac{\epsilon}{3})$ as in Proposition 5.2.*

Proof. By Proposition 5.2, diameter of Σ modulo $\Sigma_{(0,\frac{\epsilon}{3}]} < D$. Therefore $\Sigma \subset nbhd_D(\Sigma_{(0,\frac{\epsilon}{3}]})$. As f is distance-decreasing, (By Lemma 2.6.1 [16]), we have

$$f(\Sigma) \subset f(nbhd_D(\Sigma_{(0,\frac{\epsilon}{3}]})) \subset nbhd_D(f(\Sigma_{(0,\frac{\epsilon}{3}]})).$$

Let $x \in \Sigma_{(0,\frac{\epsilon}{3}]}$, γ be a closd curve passing through x such that γ is nontrivial in Σ and $length(\gamma) < \frac{\epsilon}{3}$. Then we can consider the following cases.

Case 1: $\gamma \cap S \neq \emptyset$.

Then we have $x \in nbhd_{\frac{\epsilon}{3}}(S)$. As a result, there is some i such that $f(x) \in nbhd_{\frac{\epsilon}{3}}(\gamma_i) \subset M_{(0,\epsilon]}$. Therefore $f(x) \in T_i$.

Case 2: $\gamma \cap S = \emptyset$.

Then $\gamma \subset R$ component of $\Sigma - S$ and therefore γ is nontrivial in R. (By Definition 5.10 (iv).) Therefore there exists $C' \subset \gamma$ simple closed curve which is nontrivial in R. By Definition 5.10 (ii), there exists C component of S such that C' parallel C. As $length(\gamma) < \frac{\epsilon}{3}$, we have $length(C') < \frac{\epsilon}{3}$ and $x \in nbhd_{\frac{\epsilon}{3}}(C')$. Therefore there exists loop γ' base at x, freely homotopic to C and $length(\gamma') < \frac{\epsilon}{3}$. Then $length(f(\gamma')) < \epsilon$ and freely homotopic to γ_i for some $\gamma_i = C$. Hence $f(\gamma') \subset T_i$. As a result, $f(x) \in T_i$.

Therefore $f(\Sigma) \subset nbhd_D(T_1 \cup ... \cup T_k)$ $\square$

Definition 5.11. *Let (Σ, f, S) be an ϵ-girded surface in M, $K \subset M$. (Σ, f, S) is called enclose K if (Σ, f) is null-homologous and enclose K.*

Definition 5.12. *A complete hyperbolic 3-manifold M is called girded if there exists $g \in \mathbb{Z}$, $p \in M$ such that for all $\epsilon > 0$, there exists ϵ-girded surface (Σ, f, S) of total genus $\leq g$ and (Σ, f, S) enclose p.*

Proposition 5.10. *Let M be a complete hyperbolis 3-manifold. Suppose M is girded. Then every positive superharmic function on M is constant.*

In the remaining section we will use the following notaion:

$V := PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$;

$D := \{(\xi, \eta) \in V : \langle \xi, \eta \rangle$ is a Kleinian group, free of rank $2\}$;

$NP := \{(\xi, \eta) \in D : \langle \xi, \eta \rangle$ without parabolics $\}$;

$GF := \{(\xi, \eta) \in D : \langle \xi, \eta \rangle$ is a geometrically finite group without parabolics $\}$;

$B :=$ frontier of GF in $V = \overline{GF} - GF$.

Remark. D is closed, GF is open in V. (By [12].)

Then we state some related results.

Lemma 5.1. *Let $(\xi, \eta) \in B$. Suppose (Σ, f) is a hyperbolic simplicial surface in $M(\langle \xi, \eta \rangle)$. Then (Σ, f) is null-homologous.*

Proof. We have $H_2(M(\langle\xi,\eta\rangle);\mathbb{Z}/2) \cong H_2(\langle\xi,\eta\rangle\,;\mathbb{Z}/2) = 0$ as $\langle\xi,\eta\rangle$ is free. $\square$

By Corollary 1.5 [22], we have the following propositions.

Proposition 5.11. *$NP \cap B$ is a dense G_δ set in B.*

Proposition 5.12. *There exists a C dense in B such that $\forall(\xi,\eta) \in C$, $\langle\xi,\eta\rangle$ satisfies (i)$\langle\xi,\eta\rangle$ is free Kleinian group, (ii)$\langle\xi,\eta\rangle$ is geometrically finite and (iii)$\langle\xi,\eta\rangle$ has three distinct conjugacy classes of cuspidal subgroup.*

Let $z_0 \in \mathbb{H}^3$ be chosen. Let $(\xi,\eta) \in D$, then there exists $l(\xi,\eta)$ and τ_l by Proposition 2.3. Then $z \in M(\langle\xi,\eta\rangle)$ is called a reference point if either one of the following:

(i) preimage of $z \in l(\xi,\eta)$ and closest to z_0;

(ii) preimage of $z \in \xi(l(\xi,\eta))$ and closest to z_0;

(iii) preimage of $z \in \eta(l(\xi,\eta))$ and closest to z_0.

Proposition 5.13. *Let $(a,b) \in D$ such that $\langle a,b\rangle$ satisfies (ii),(iii) of Proposition 5.12. Let $\epsilon > 0$. Then there exists $U = U_\epsilon(a,b) \subset B$ neighborhood of (a,b) such that $\forall(\xi,\eta) \in NP \cap U$, there exists a connected ϵ-girded surface of genus 2 in $M(\langle\xi,\eta\rangle)$ which encloses at least one reference point of $M(\langle\xi,\eta\rangle)$.*

The main result of this section is:

Theorem 5.1. *There exists a dense G_δ-set C in B such that for all $(\xi,\eta) \in C$, $\langle\xi,\eta\rangle$ is girded.*

Proof. By Proposition 5.12 and 5.13, there exists $W_\epsilon = \bigcup_{(a,b)\in C} U_\epsilon(a,b)$ open and dense in B, $\forall \epsilon > 0$. Then by Proposition 5.11, the set

$$C' = NP \cap B \cap (\bigcap_{n\in\mathbb{N}} W_{\frac{1}{n}}) = NP \cap (\bigcap_{n\in\mathbb{N}} W_{\frac{1}{n}})$$

satisfies the requirment. $\square$

Chapter 6

Margulis number of Hyperbolic Manifolds

In this chapter, we study about how Theorem 6.1 and our main result Theorem 6.3 can be proved by using the previous results. As our concerning case is closed, orientable hyperbolic 3-manifold, we only have to consider torsion free discrete group without parabolics.

6.1 Geomertrically finite groups

In this section, we mainly deal with 2-generating torsion free discrete group without parabolics and prove some corresponding result.

By Proposition 3 and Theorem A [5] and p.59 [20], we have:

Proposition 6.1. *Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$ with $\xi\eta \neq \eta\xi$. Suppose $\langle \xi, \eta \rangle$ is torsion free, discrete, topologically tame, noncocompact and contains no parabolics. Then $\langle \xi, \eta \rangle$ is a free group on ξ and η.*

Proposition 6.2. *Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$ with $\xi\eta \neq \eta\xi$. Suppose $\langle \xi, \eta \rangle$ is torsion free, discrete, topologically tame, noncocompact and contains no parabolics. Then either $\langle \xi, \eta \rangle$ is geometrically finite or $M(\langle \xi, \eta \rangle) = N(\langle \xi, \eta \rangle)$.*

Lemma 6.1. *Let Γ be a Kleinian group contains no parabolics and is free on generators ξ, η. Suppose Γ is topologically tame and $M(\Gamma) = N(\Gamma)$. Then* $\max(d(z, \xi(z)), d(z, \eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$.

Proof. By Proposition 5.6, we have Proposition 3.2 holds and hence Theorem 4.3 valid. Therefore $\max(d(z, \xi(z)), d(z, \eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$. □

Lemma 6.2. *Let Γ be a Kleinian group contains no parabolics and is free on generators ξ, η. Suppose Γ is geometrically finite. Then* $\max(d(z, \xi(z)), d(z, \eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$.

Proof. Let $z \in \mathbb{H}^3$. Define $f : V \to \mathbb{R}$ by $f(\xi, \eta) = \max(d(z, \xi(z)), d(z, \eta(z)))$. Then f is continuous and our aim is to show that $f \geq \log 3$ on $\overline{GF}$. Firstly, by dicreteness, we can show that mimimum of f is not occur on GF. Then we consider the dense set C in B from Theorem 5.1. As $\langle \xi, \eta \rangle$ is girded, $\forall (\xi, \eta) \in C$, we know that from Proposition 5.10, Theorem 4.3 is valid. As a result, we have $\min_{(\xi,\eta)\in B} f \geq \log 3$ as f is continuous. Therefore $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$. □

The main theorem of this section is:

Theorem 6.1. *Let $\xi, \eta \in Isom^+(\mathbb{H}^3)$ with $\xi\eta \neq \eta\xi$. Suppose $\langle \xi, \eta \rangle$ is torsion free, discrete, topologically tame, noncocompact and contains no parabolics. Then* $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$, $\forall z \in \mathbb{H}^3$.

Proof. By Proposition 6.1 and 6.2, we have $\langle \xi, \eta \rangle$ satisfies either Lemma 6.1 or 6.2 and both cases lead to the conclusion $\max(d(z,\xi(z)), d(z,\eta(z))) \geq \log 3$. □

6.2 Margulis number of Closed Hyperbolic Manifolds

In this last section, we illustrate how the main theorem can be proved.

Theorem 6.2. *Let $M(\Gamma)$ be a closed, orientable hyperbolic 3-manifold such that $rank(H_1(M;\mathbb{Q})) \geq 3$. Then every 2-generator subgroup of Γ is noncocompact and topologically tame.*

Proof. Let $\langle \xi, \eta \rangle \leq \Gamma$ be a 2-generator subgroup. As $rank(H_1(M;\mathbb{Q})) \geq 3$, we know that $\langle \xi, \eta \rangle$ has infinite index. Therefore $M(\langle \xi, \eta \rangle)$ is not compact. As $\langle \xi, \eta \rangle$ is finitely generated, it must be topologically tame. (By Theorem 10.2 [2].) □

Theorem 6.3. *Let M be a closed, orientable hyperbolic 3-manifold such that $rank(H_1(M;\mathbb{Q})) \geq 3$. Then* $\log 3$ *is a Margulis number for M.*

Proof. As the assumptions of Theorem 6.2 hold, we may apply it and hence find that the assumptions of Theorem 6.1 also holds because the thin part does not contain cusps as M is compact. Then we can apply Theorem 6.1 and get the required result. □

As the Tameness Theorem is proved by Agol in 2004, we can get a further result according to Shalen and Culler that $\log 3$ is a Margulis number for all closed, orientable hyperbolic 3-manifold whose fundamental group has no 2-generator subgroup of finite index.

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